Mathematical Approach to Current Sharing Problem of Superconducting Triple Strands

Hiroaki Tsutsui, Shinichi Nomura, Ryuichi Shimada, and Shunji Tsuji-Iio

Abstract— **The current sharing between insulated strands in a superconducting cable is one of the important problems for its utilization. From the view points of the inverse problem, the sensitivity of current sharing between the insulated strands is determined by the condition number of the inductance matrix. For the triple strands with the self similar structure, we derive the analytic form of the inductance matrix which only includes two parameters; the self inductance of a unit wire, the ratio of mutual to self inductance for unit wires. Since the matrix elements also have the self similar structure, we can analytically obtain the eigenvalues, eigenvectors and condition number, which is the ratio of maximum and minimum eigenvalues. Next, we derive the formula to estimate the sensitivity of the current distribution against the displacement of inductance from the ideal case by use of the condition number. This formula shows that the sensitivity is inversely proportional to the difference of self and mutual inductances of unit wires. Moreover, we estimate the condition number of the very thin wire to check our formula. Finally, we verify our analytic form by numerical calculations.**

Keywords— **SMES, superconductor, strand, current sharing, fractal, self similar, inverse problem, inductance**

I. Introduction

THE progress of the superconducting material technology in recent years makes possible to produce the magnet made of superconductor, and the superconducting magnetic energy storage (SMES)[1]-[3] is also studied. In the field of the nuclear fusion, the superconducting technology is realized as a coil of the Large Helical Device[4] and ITER[5]. One of the important problems in making the cable from the superconducting wire is the current sharing phenomenon[6]-[8]. To use the current capacity of the coil fully, it is necessary that the current uniformly flows in all the filaments that compose the coil. However, the current distribution is determined by the inductance because the superconducting wire has no electric resistance. Therefore, almost all superconducting wires consist of many stranded filaments to make inductances of the filaments uniform. However, even if such a wire is utilized, it is difficult to remove the dispersion of the inductance, and they can use only about 30% of the current capacity for the present.

In this work, we investigate the current sharing phenomenon of triple strands and derive the theoretical formula which shows the degree of sharing. In the next section, we derive the basic equations of current sharing phenomenon and explain the principle of the phenomenon. In section III, we analyze the current sharing of the simple triple strands. In section IV, we extend the analysis to the current sharing of the multiple triple strands, and derive the degree of sharing quantitatively. In the last section, we summarize this study.

Manuscript received September 24, 2001.
H. Tsutsui, S. Nomura, R. Shimada and S. Tsuji-Iio are with Research Laboratory for Nuclear Reactors, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo 152-8550, Japan (telephone: +81-3-5734-3839, e-mail: htsutsui@nr.titech.ac.jp).

II. Basic Equations

When the electric resistivity is negligible, the current flowing in the circuit follows the next equation,

$$\mathsf{L}\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \boldsymbol{V}, \tag{1}$$

where L is the inductance matrix of the element lines, $\boldsymbol{I}$ is the vector of current flowing in each element line, $\boldsymbol{V}$ is the vector of volage added to each element line. Because we add usually the same voltage to each element line, (1) is reduced to

$$\begin{aligned} \mathsf{L}\boldsymbol{I} &= \Phi\boldsymbol{u}, \qquad (2)\\ \boldsymbol{u} &\equiv \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \\ \Phi &\equiv \int V(t)\mathrm{d}t. \end{aligned}$$

This is the basic equation for the current sharing of superconducting strands. Although Φ is the function of time in general, we assume Φ is constant in this work because we need the ratio of current only.

First of all, consider the simplest strands of two lines to understand the current sharing. In this case, (2) is reduced to

$$\begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix}\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \Phi\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3}$$

Therefore, the current ratio of I_1/I_2 is

$$\frac{I_1}{I_2} = \frac{L_2 - M}{L_1 - M}. \tag{4}$$

This equation shows that the source of sharing is the dispersion of the self inductance. When the mutual inductance M is much smaller than the self inductance L, the current sharing is small even though the values of the self inductance are dispersed. Unfortunately, the small dispersion of L is amplified in the case of strands because of $M \simeq L$.

III. Simple Triple Strands

In this section, we consider the case of simple triple strands. Because the magnetic coupling between each element line is strong in the case of the strands, we assume that mutual inductance is barely smaller in comparison with self inductance. By use of small positive parameter ϵ_i, the normalized inductance matrix can be reduced to

$$\mathsf{L} = \begin{pmatrix} 1+\epsilon_1 & 1 & 1 \\ 1 & 1+\epsilon_1 & 1 \\ 1 & 1 & 1+\epsilon_2 \end{pmatrix}. \qquad (5)$$

Here, we assume that all components of mutual inductance have the same value and one of the self inductances is changed barely. In this case, the solution to current vector is

$$\boldsymbol{I} = \begin{pmatrix} 1 \\ 1 \\ \frac{\epsilon_1}{\epsilon_2} \end{pmatrix}. \qquad (6)$$

This solution shows all the components of current, which are distributed, have the same direction. Next, we distribute the values of mutual inductances while the value of self inductance is fixed. The inductance matrix in this case is as follows:

$$\mathsf{L} = \begin{pmatrix} 1 & 1-\epsilon_1 & 1-\epsilon_2 \\ 1-\epsilon_1 & 1 & 1-\epsilon_1 \\ 1-\epsilon_2 & 1-\epsilon_1 & 1 \end{pmatrix}. \qquad (7)$$

The solution to current vector for the inductance matrix is

$$\boldsymbol{I} = \begin{pmatrix} 1 \\ 2-\frac{\epsilon_2}{\epsilon_1} \\ 1 \end{pmatrix}. \qquad (8)$$

This solution shows that the large dispersion of mutual inductance reverses the direction of current in a filament. Therefore, it is expected that the distribution of current sharing strongly depends on the dispersion of mutual inductance.

Since the values of each element of the inductance matrix of the strands are close to each other, the solutions to (5) and (7) are undetermined in the case of $\epsilon_1 = \epsilon_2 = 0$ because the determinant of the matrixes is zero. The eigenvalues and eigenvectors of L in the case of $\epsilon_1 = \epsilon_2 = 0$ are as follows:

$$\lambda_1 = 3, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \qquad (9)$$

$$\boldsymbol{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{x}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{x}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \qquad (10)$$

Using arbitrary real parameters of a_2, a_3, the solution to this equation is expressed as

$$\boldsymbol{I} = \boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + a_3\boldsymbol{x}_3. \qquad (11)$$

All components of the current vector has the same value in the case of $a_2 = a_3 = 0$ only.

Next, we consider the more general problem. We expand the current vector $\boldsymbol{I}$ and the constant vector $\boldsymbol{u}$ in the right-hand side of (2) by the eigenvectors $\boldsymbol{x}_i$ of the inductance matrix L as follows:

$$\boldsymbol{I} = \sum_i a_i\boldsymbol{x}_i, \qquad \boldsymbol{u} = \sum_i b_i\boldsymbol{x}_i. \qquad (12)$$

When these equations are substituted to (2), the next equation is obtained,

$$\sum_i a_i\lambda_i\boldsymbol{x}_i = \sum_i b_i\boldsymbol{x}_i, \qquad (13)$$

where λ_i are the eigenvalues of L. Since the inductance matrix is positive definite and symmetric, the eigenvectors are orthogonal to each other. Hence, the coefficients are determined as follows:

$$a_i = \frac{b_i}{\lambda_i}. \qquad (14)$$

This equation shows that the components with small eigenvalue are amplified. Therefore, the solution to the simultaneous linear equations by matrix with small eigenvalues are sensitive to the change in coefficients, and called as ill-posed problem. Parameter that shows this sensitive nature is called the condition number[9] and is defined by the following equation in this problem.

$$\mathrm{cond}(\mathsf{L}) = ||\mathsf{L}|| \cdot ||\mathsf{L}^{-1}|| = \frac{\lambda_{\max}}{\lambda_{\min}} \qquad (15)$$

Here, $||\mathsf{L}||$ is norm of matrix L defined as the maximum eigenvalue. The second equality derived by use of the relation that the maximum eigenvalue of L^{-1} is the minimum eigenvalue of L for the positive definite matrix. Using the condition number, the change in current caused by the change in inductance is represented as,

$$\frac{|\delta\boldsymbol{I}|}{|\boldsymbol{I}|} \le \mathrm{cond}(\mathsf{L})\frac{||\delta\mathsf{L}||}{||\mathsf{L}||} = \frac{||\delta\mathsf{L}||}{\lambda_{\min}}. \qquad (16)$$

When we apply this result to the inductance matrix of (7), we get the next equation,

$$\frac{|\delta\boldsymbol{I}|}{|\boldsymbol{I}|} \le \frac{||\delta\mathsf{L}||}{\epsilon} = \frac{\xi}{\epsilon}, \qquad (17)$$

where $\epsilon \equiv \epsilon_1$, $\xi \equiv |\epsilon_1 - \epsilon_2|$. Since the relation of $|\delta\boldsymbol{I}|/|\boldsymbol{I}| = \xi/\epsilon$ holds in (8), (16) expresses the degree of sharing of an element line current very precisely in the case of the simple triple strands.

IV. Multiple Triple Strands

In this section, we consider the multiple triple strands which is multiply combined by many simple triple strands. In Fig. 1 the cross section of it is represented. Here, we define the layers as follows: the element wire is level 0, the first triple strand which consists of three wires is level 1, the second level strand consists of the first level strand, and so on. Because the strand of level n consists of the strands of level $n-1$, the physical quantities on level n can be represented by those of subscript n. The relation of inductances on level n and $n-1$ is expressed as

$$9L_n = 3L_{n-1} + 6M_{n-1}. \qquad (18)$$

Here we follow the example of the previous section and set mutual inductance of each level as

$$M_n = (1-\epsilon_n)L_n \qquad (0 < \epsilon_n < 1). \qquad (19)$$

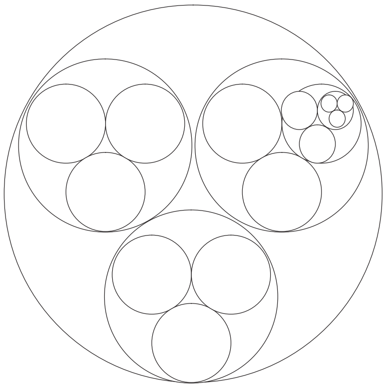

Fig. 1. Schematic cross section of multiple triple strand.

This is also the definition of ϵ_n, which relates to the magnetic coupling of wires. When $\epsilon_n = 1$, each wire is independent to other wires, while all wires are perfectly coupled to each other in the case of $\epsilon_n = 0$. Substituting this equation to (18), The next relation is derived,

$$\frac{L_n}{L_{n-1}} = 1 - \frac{2}{3}\epsilon_{n-1} \equiv \sigma_{n-1}. \tag{20}$$

Using this equation, we get the inductance on each level as follows:

$$\begin{aligned} L_n &= \sigma_{n-1}L_{n-1} = \cdots = L_0\Pi_{i=0}^{n-1}\sigma_i, &(21)\\ M_n &= (1-\epsilon_n)L_n = L_0(1-\epsilon_n)\Pi_{i=0}^{n-1}\sigma_i. &(22)\end{aligned}$$

On the other hand, the inductance matrix on level 1 is represented as

$$\mathsf{L}_1 = \begin{pmatrix} L_0 & M_0 & M_0 \\ M_0 & L_0 & M_0 \\ M_0 & M_0 & L_0 \end{pmatrix}. \tag{23}$$

Using the self similarity of the multiple triple strands shown in Fig. 1, the inductance on level n is expressed as

$$\begin{aligned} \mathsf{L}_n &= \begin{pmatrix} \mathsf{L}_{n-1} & \mathsf{M}_{n-1} & \mathsf{M}_{n-1} \\ \mathsf{M}_{n-1} & \mathsf{L}_{n-1} & \mathsf{M}_{n-1} \\ \mathsf{M}_{n-1} & \mathsf{M}_{n-1} & \mathsf{L}_{n-1} \end{pmatrix}, &(24)\\ \mathsf{M}_{n-1} &= \begin{pmatrix} M_{n-1} & \cdots \\ \vdots & \ddots \end{pmatrix}. &(25)\end{aligned}$$

In this case, the maximum and the minimum eigenvalues are evaluated by the use of (22) as

$$\lambda_{\max} = L_0 3^n \Pi_{i=0}^{n-1}\sigma_i, \quad \lambda_{\min} = L_0\epsilon_0. \tag{26}$$

Since the eigenvector with the maximum eigenvalue is $\boldsymbol{u}$, the solution to (2) for the multiple triple strands is the current vector whose all components have the same value. However, because the components of the actual inductance matrix are distributed, the degree of dispersion for the current of each filament against the distribution of the inductance matrix $||\delta\mathsf{L}||$ is represented by use of (16) as

$$\frac{|\delta \boldsymbol{I}|}{|\boldsymbol{I}|} \leq \frac{3^n\Pi_{i=0}^{n-1}\sigma_i}{\epsilon_0}\frac{||\delta\mathsf{L}_n||}{||\mathsf{L}_n||} = \frac{||\delta\mathsf{L}_n||}{\epsilon_0 L_0}. \tag{27}$$

This equation is a general form of current sharing in the multiple triple strands. It shows that the current sharing increases with the strength of the magnetic coupling between the filaments $1/\epsilon_0$ and the number of nesting level n.

To evaluate (27), we need the expression of $||\delta\mathsf{L}_n||$. When the deviation of inductance for each filament is almost equivalent, we can put the error matrix $\delta\mathsf{L}_0$ of L_0 as follows:

$$\delta\mathsf{L}_0 = \xi L_0 \begin{pmatrix} 1 & -1 & \cdots & -1 \\ -1 & 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & 1 \end{pmatrix},$$

where ξ represents the degree of dispersion of inductance. Since the relation $||\delta\mathsf{L}_n|| = 3^n\xi L_0$ holds in this case, we get the next relation,

$$\frac{|\delta \boldsymbol{I}|}{|\boldsymbol{I}|} \leq 3^n\frac{\xi}{\epsilon_0}. \tag{28}$$

It is also clear that the current sharing increases with the coupling between the filaments $1/\epsilon_0$ and the number of nesting level n.

Next, we apply the result to the ring-shaped coil of radius R. Since the pitch of winding of elementary filaments is small in general, the self inductance and mutual inductance of the filament with radius a are represented as

$$\begin{aligned} L_0 &= \mu_0 R\left(\log\frac{8R}{a} - \frac{7}{4}\right), \\ M_0 &= \mu_0 R\left(\log\frac{8R}{2a} - 2\right). \end{aligned}$$

For example, we employ the Nb-Ti filament with a radius $a = 1\mu$m which is used for the helical coil of LHD[4]. When we put the radius of the coil $R = 1$m, the coupling parameter ϵ_0 of the filaments is evaluated as follows:

$$\epsilon_0 = \frac{L_0 - M_0}{L_0} = \frac{\log 2 + \frac{1}{4}}{\log\frac{8R}{a} - \frac{7}{4}} \simeq 0.07.$$

In this example, the condition $\xi \simeq 1 \times 10^{-4}$ is required to suppress the error of current less than 10% in the case of 4 layers.

V. Summary and Discussions

To investigate the current distribution of strands, the inductance matrix is analyzed using the technique of the inverse problem. For the simple triple strands, the reversal of the current is caused by the dispersion of mutual inductance.

Furthermore, the inductance matrix of the multiple triple strands is analytically derived using the self similarity of the strands. This alalytic form includes only two parameters; the self inductance of a unit wire, the ratio of mutual to self inductance for unit wires. The degree of current sharing against the dispersion of components of the inductance matrix is also obtained analytically. According to the analytic form of the degree of current sharing, the degree of sharing increases with the number of layers and the magnetic coupling between the filaments. Usually, the degree of sharing is evaluated numerically, but the calculation is unstable because of the singularity of the inductance matrix. Since our formula utilizes the singularity, it enables us to evaluate the degree of sharing precisely.

In this work, we restrict the types of strands to the triple strands for simplicity. Since the method in this work is easily extended to the other multiple strands, the results of this work is also applicable to general superconducting coils.

References

[1] Y.Sato, *et. al.*: *Proceedings of 15th International Conference on Magnet Technology (MT15)* 542, Oct. 20-24, 1997, Beijing, China.

[2] S. Nomura, T. Osaki, J. Kondoh, H. Tsutsui, S. Tsuji-Iio, Y. Sato and R. Shimada: IEEE Tran. Appl. Superconductivity, **9**(2), (1999) 354-357.

[3] S. Nomura, *et al.*: IEEE Tran. Appl. Superconductivity **11**(1), 1920 (2001)

[4] K. Kitamura, *et al.*: Fusion Eng. Des., **20**, 167 (1993)

[5] C. Sborchia, *et al.*: IEEE Tran. Appl. Superconductivity, **10**, No.1, 554 (2000)

[6] R. Schermer, B. Turck: Adv. Cryog. Eng., **26**, 599 (1980)

[7] N. Koizumi, K. Okuno, Y. Takahashi, H. Tsuji, S. Shjmamoto: Cryogenics, **36**, 409 (1996)

[8] S. Yamaguchi, J. Yamamoto, O. Motojima: Cryogenics, **36**, 661 (1996)

[9] A.S. Householder: "The Theory of Matrices in Numerical Analysis", Dover (1975)